International Journal of Theoretical Physics, Vol. 25, No. 4, 1986

New Approach to Einstein–Petrov Type I Spaces. II. A Classification Scheme

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Received November 11, 1985

A classification scheme is presented which enables one to have an overall look at Einstein-Petrov type I spaces and in particular to pick out those spaces whose Petrov scalars are functionally independent. It is shown that some subclasses defined by quite severe simplifying assumptions (which lead to considerable simplification of the field equations) still retain functionally independent Petrov scalars. For these subclasses three coupled differential equations (two of which are wave equations) are calculated for the Petrov scalars.

1. INTRODUCTION

The distinction between algebraically general and algebraically special spaces and the subsequent subdivision of the algebraically special spaces initiated a systematic search for exact vacuum solutions of the algebraically special equations; this has been very successful. This distinction between algebraically special and algebraically general spaces has of course important physical significance; but from the purely computational viewpoint, the imposition of the algebraically special restrictions reduces the large number of unwieldy field equations in arbitrary space to a more manageable number, especially when the formalism of Newman and Penrose (1962) (NP) is used. Indeed, the NP formalism is ideally suited to investigating algebraically special spaces since the restriction to these spaces can be imposed in a natural way, as can the additional restrictions to each of the various subclasses of algebraically special spaces.

There has been very little success in finding exact solutions to the algebraically general vacuum equations; in fact it has not even been possible to make any natural subdivision of the algebraically general spaces that could form the basis for a systematic approach. An examination of the NP

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0020-7748/86/0400-0425$05.00/0

equations for algebraically general spaces does not suggest any obvious way—even from purely computational considerations—of picking out a particular subclass of spaces with simplified and more manageable equations. Indeed, it would seem that the NP formalism is not particularly suited to the search for exact vacuum solutions of the algebraically general equations.

In a previous paper (Edgar, 1979; henceforth referred to as I) I proposed a modification of the NP formalism that is specially designed for algebraically general vacuum spaces—in particular, for those spaces whose complex Petrov scalars supply four functionally independent real functions. These spaces—the most "general" of the algebraically general—are the spaces we would expect to be fully representative of their class. Associated with this formalism is an integration procedure built on scalar wave equations for the Petrov scalars, or other scalar functions.

Within this large class of spaces one would like some sort of systematic classification so that one could concentrate initially on some spaces that are defined by some simplifying assumptions giving simplified field equations, and gradually move on to spaces with less and less simplifications imposed on the field equations. In this paper I show that such a classification scheme suggests itself quite naturally in the modified formalism of I when conditions are examined for the two complex Petrov scalars to be functionally independent.

The modified formalism is summarized in Section 2, and the classification scheme is presented in Section 3. As one would expect, the functionally dependent restriction on the Petrov scalars usually simplifies the field equations considerably, while the functionally independent restriction, in general, leaves us still with complicated field equations; however, what is most interesting about the classification scheme is that it picks out some spaces whose field equations are comparatively simple, but whose Petrov scalars are still functionally independent.

Sections 4 and 5 look at these spaces in detail, showing how their field equations yield the scalar wave equations that make them eligible for the integration procedure outlined in I. Section 6 summarizes the results.

2. THE FIELD EQUATIONS

The basic system of equations usually considered in the tetrad formalism is the first set of structure equations, the second set of structure equations, and the Bianchi equations. The NP formalism sets out each individual equation explicitly.[2] The modifications made here are as follows:

[2]The standard NP formalism of Newman and Penrose (1962) will be assumed.

1. The tetrad freedom is used to put the Weyl tensor into canonical form,

$$\Psi_1 = 0 = \Psi_3, \qquad \Psi_0 = \Psi_4 \tag{1}$$

2. The spin coefficients are relabeled,[3]

$$A_m = (\rho, -\mu, \tau, -\pi), \qquad B_m = (-\lambda, \sigma, -\nu, \kappa), \qquad C_m(-\varepsilon, \gamma, -\beta, \alpha) \tag{2}$$

The first set of structure equations are written as

$$\begin{aligned}
A_\lambda &= Z_{2\lambda} Z_{1\mu;\nu} Z_3{}^\mu Z_4{}^\nu + Z_{1\lambda} Z_{2\mu;\nu} Z_4{}^\mu Z_3{}^\nu \\
&\quad - Z_{4\lambda} Z_{1\mu;\nu} Z_3{}^\mu Z_2{}^\nu - Z_{3\lambda} Z_{2\mu;\nu} Z_4{}^\mu Z_1{}^\nu \\
B_\lambda &= Z_{2\lambda} Z_{2\mu;\nu} Z_4{}^\mu Z_4{}^\nu + Z_{1\lambda} Z_{1\mu;\nu} Z_3{}^\mu Z_3{}^\nu \\
&\quad - Z_{4\lambda} Z_{2\mu;\nu} Z_4{}^\mu Z_2{}^\nu - Z_{3\lambda} Z_{1\mu;\nu} Z_3{}^\mu Z_1{}^\nu \\
2C_\lambda &= Z_{2\lambda}(Z_{2\mu;\nu} Z_1{}^\mu Z_1{}^\nu + Z_{3\mu;\nu} Z_4{}^\mu Z_1{}^\nu) \\
&\quad + Z_{1\lambda}(Z_{1\mu;\nu} Z_2{}^\mu Z_2{}^\nu + Z_{4\mu;\nu} Z_3{}^\mu Z_2{}^\nu) \\
&\quad - Z_{4\lambda}(Z_{2\mu;\nu} Z_1{}^\mu Z_3{}^\nu + Z_{3\mu;\nu} Z_4{}^\mu Z_3{}^\nu) \\
&\quad - Z_{3\lambda}(Z_{1\mu;\nu} Z_2{}^\mu Z_4{}^\nu + Z_{4\mu;\nu} Z_3{}^\mu Z_4{}^\nu)
\end{aligned} \tag{3}$$

Three of the second set of structure equations became

$$\begin{aligned}
A^\mu{}_{;\mu} + A^\mu A_\mu - B^\mu B_\mu &= -2\Psi_2 \\
B^\mu{}_{;\mu} + 4C_\mu B^\mu &= 2\Psi_0 \\
C^\mu{}_{;\mu} + \tfrac{1}{2}B^\mu B_\mu - \tfrac{1}{2}A^\mu A_\mu + 2C^\mu A_\mu &= 2\Psi_2
\end{aligned} \tag{4}$$

and the Bianchi equations become

$$\begin{aligned}
\Psi_{2,\mu} - 3A_\mu \Psi_2 - B_\mu \Psi_0 &= 0 \\
\Psi_{0,\mu} - 3B_\mu \Psi_2 - (4C_\mu + A_\mu)\Psi_0 &= 0
\end{aligned} \tag{5}$$

Because of the considerable redundancy in the original basic system, it has been shown in I that the particular subset of equations (3)-(5) is itself a sufficient subsystem for algebraically general vacuum spaces, provided that Ψ_0 and Ψ_2 supply four functionally independent real functions. This subset is not sufficient for cases where Ψ_0 and Ψ_2 supply less than four independent real functions; however, it is still a necessary subsystem for such spaces.

[3]The orthonormal tetrad vectors are denoted by $\mathbf{Z}_m{}^\mu$ and the tetrad components of the Riemann tensor by $\mathbf{R}_{mnrq}$. In general, the letters in the latter half of the Latin alphabet, $m, n, p, \ldots$, will be used for tetrad components of an object, while the letters in the latter half of the Greek alphabet, $\rho, \lambda, \mu, \ldots$, will be used for coordinate components. All indices run 1, 2, 3, 4. The covariant derivative and the intrinsic derivative will be denoted by a semicolon, while the ordinary partial derivative will be denoted by a comma. Antisymmetrization will be denoted by square brackets.

Brans (1977) has pointed out that when the Weyl tensor is in canonical form, the Bianchi equations have nontrivial integrability conditions. In I it was shown that these post-Bianchi equations can be stated very concisely as follows:

$$\begin{aligned} 3\Psi_2\{A_{[\mu;\nu]}\}+\Psi_0\{B_{[\mu;\nu]}+2A_{[\mu}B_{\nu]}+4B_{[\mu}C_{\nu]}\}=0 \\ 3\Psi_2\{B_{[\mu;\nu]}-2A_{[\mu}B_{\nu]}-4B_{[\mu}C_{\nu]}\}+\Psi_0\{A_{[\mu;\nu]}+4C_{[\mu;\nu]}\}=0 \end{aligned} \tag{6}$$

It should be pointed out that in I it was claimed that the set of equations (4)-(6) is a sufficient system of equations, i.e., the rather awkward set (3) is satisfied identically if the other three sets are satisfied. In fact this was not proved in I as implied, and although there are strong indications that it is true, a general proof has proved elusive. At this stage I include the set of equations (3) as an essential component of the sufficient system; however, my ultimate aim is to integrate equations (4) and (5) to obtain a metric tensor and then to check it by computing techniques, rather than tackle (3) directly.

3. CLASSIFICATION OF ALGEBRAICALLY GENERAL VACUUM SPACES

One of the motivations for the present classification scheme is to be able to pick out those spaces whose four real Petrov scalars are functionally independent. However, since we are dealing with the two complex Petrov scalars Ψ_0 and Ψ_2, it will be convenient to base the classification scheme on them; it will be easy to impose the additional conditions on the real and imaginary parts when required.

The theorem that motivates the present classification is:

Theorem 1. Ψ_0 and Ψ_2 are functionally dependent if, and only if, for A_μ, B_μ, C_μ all nonzero:

(a) A_μ, B_μ, C_μ are parallel vectors; or
(b) A_μ, B_μ, C_μ are linearly dependent, i.e.,

$$pA_\mu+qB_\mu+rC_\mu=0$$

(with p, q, r not all zero) and

$$r(9\Psi_2{}^2-\Psi_0{}^2)-12q\Psi_0\Psi_2+4p\Psi_0{}^2=0$$

Proof. A necessary and sufficient condition for Ψ_0 and Ψ_2 to be functionally dependent is

$$\Psi_{2,[\mu}\Psi_{0,\nu]}=0 \tag{7}$$

which is equivalent, from (5), to

$$(9\Psi_2{}^2-\Psi_0{}^2)A_{[\mu}B_{\nu]}+12\Psi_0\Psi_2A_{[\mu}C_{\nu]}-4\Psi_0{}^2C_{[\mu}B_{\nu]}=0 \tag{8}$$

From (8) it follows that

$$A_{[\mu}B_\nu C_{\lambda]}=0 \tag{9}$$

Thus, substituting

$$pA_\mu+qB_\mu+rC_\mu=0 \tag{10}$$

in (8) gives the necessary conditions of the theorem. Conversely, if the conditions (a) and (b) of the theorem are satisfied, (8) is satisfied. ■

The following classification is proposed based on the relationship between the three vectors A_μ, B_μ, C_μ:

Class A. At least one of the three vectors A_μ, B_μ, C_μ is zero.

Class B. (a) The three nonzero vectors A_μ, B_μ, C_μ are all parallel.

(b) The three nonzero vectors A_μ, B_μ, C_μ are linearly dependent, i.e.,

$$pA_\mu+qB_\mu+rC_\mu=0$$

(with p, q, r not all zero) and

$$4p\Psi_0{}^2-12q\Psi_0\Psi_2+r(9\Psi_2{}^2-\Psi_0{}^2)=0$$

Class C. The three nonzero vectors A_μ, B_μ, C_μ, are linearly dependent (but not all parallel), i.e.,

$$pA_\mu+qB_\mu+rC_\mu=0$$

(with p, q, r not all zero) and

$$4p\Psi_0{}^2-12q\Psi_0\Psi_2+r(9\Psi_2{}^2-\Psi_0{}^2)\neq 0$$

Class D. The three nonzero vectors A_μ, B_μ, C_μ are linearly independent.

From Theorem 1 two corollaries follow immediately:

Corollary 1.1. For class B spaces, Ψ_0 and Ψ_2 are functionally dependent.

Corollary 1.2. For class C and class D spaces, Ψ_0 and Ψ_2 are functionally independent.

4. CLASS A SPACES

Further subdivide class A as follows:

(a) $B_\mu=0$.
(b) $A_\mu=0=C_\mu$; $B_\mu\neq 0$.
(c) $A_\mu=0$; B_μ, C_μ parallel.
(d) $C_\mu=0$; A_μ, B_μ parallel.
(e) $A_\mu=0$; B_μ, C_μ nonparallel.
(f) $C_\mu=0$; A_μ, B_μ nonparallel.

Then one can easily deduce the following;

Theorem 2.

(a) There are no class A(a) and A(b) spaces of the algebraically general type.

(b) In class A(c) and A(d) spaces, Ψ_0 and Ψ_2 are functionally dependent.

(c) In class A(e) and A(f) spaces, Ψ_0 and Ψ_2 are functionally independent.

Our interest therefore is directed toward classes A(e) and A(f). Direct substitution of the restrictions in (5) gives expressions for the two remaining nonzero vectors from A_μ, B_μ, C_μ in terms of Ψ_0 and Ψ_2, and using (4), we obtain two scalar wave equations coupled with another scalar equation for Ψ_0 and Ψ_2.

Class A(e) Spaces. Substitution of the restriction to these spaces in (5) gives

$$B_\mu = (1/\Psi_0)\Psi_{2,\mu} \tag{11}$$

$$4C_\mu = (1/\Psi_0)\Psi_{0,\mu} - (3\Psi_2/\Psi_0^2)\Psi_{2,\mu} \tag{12}$$

and from (4)

$$\Psi_{2,\mu}\Psi_2{}^{,\mu} = 2\Psi_2\Psi_0^2 \tag{13}$$

$$\Psi_2{}^{;\mu}{}_{\mu} = 6\Psi_2^2 - 2\Psi_0^2 \tag{14}$$

$$\begin{aligned}\Psi_0{}^{;\mu}{}_{\mu} &- (1/\Psi_0)\Psi_0{}^{,\mu}\Psi_{0,\mu} + (6\Psi_2/\Psi_0^2)\Psi_0{}^{,\mu}\Psi_{2,\mu} \\ &= 16\Psi_0\Psi_2 + 18\Psi_2^3/\Psi_0\end{aligned} \tag{15}$$

Class A(f) Spaces. Substitution of the restriction to these spaces in (5) gives

$$A_\mu = [1/(9\Psi_2^2 - \Psi_0^2)](3\Psi_2\Psi_{2,\mu} - \Psi_0\Psi_{0,\mu}) \tag{16}$$

$$B_\mu = [1/(9\Psi_2^2 - \Psi_0^2)](3\Psi_2\Psi_{0,\mu} - \Psi_0\Psi_{2,\mu}) \tag{17}$$

and from (4)

$$\Psi_{0,\mu}\Psi_0{}^{,\mu} - \Psi_2{}^{,\mu}\Psi_{2,\mu} = 4\Psi_0(9\Psi_2^2 - \Psi_0^2) \tag{18}$$

$$\begin{aligned}\Psi_2{}^{;\mu}{}_{\mu} &- [12\Psi_2/(9\Psi_2^2 - \Psi_0^2)]\Psi_2{}^{,\mu}\Psi_{2,\mu} \\ &+ [4\Psi_0/(9\Psi_2^2 - \Psi_0^2)]\Psi_2{}^{,\mu}\Psi_{0,\mu} \\ &= 2\Psi_0^2 + 12\Psi_0\Psi_2 + 6\Psi_2^2\end{aligned} \tag{19}$$

$$\begin{aligned}\Psi_0{}^{;\mu}{}_{\mu} &+ [4\Psi_0/(9\Psi_2^2 - \Psi_0^2)]\Psi_2{}^{,\mu}\Psi_{2,\mu} \\ &+ [12\Psi_2/(9\Psi_2^2 - \Psi_0^2)]\Psi_2{}^{,\mu}\Psi_{0,\mu} \\ &= 8\Psi_0\Psi_2 - 4\Psi_0^2\end{aligned} \tag{20}$$

Note that

$$9\Psi_2{}^2 - \Psi_0{}^2 = 0 \tag{21}$$

is a condition, in our tetrad choice, that the space be algebraically special (Penrose, 1960), and so throughout all of this work it is assumed that

$$9\Psi_2{}^2 - \Psi_0{}^2 \neq 0 \tag{22}$$

5. CLASS C SPACES

There is a very natural subdivision of these spaces:

(a) $A_\mu + kB_\mu - 2C_\mu = 0$, where k is constant.
(b) $pA_\mu + qB_\mu + rC_\mu = 0$, where p, q, r are analytic functions of Ψ_0, Ψ_2.
(c) Other than (a) and (b).

These subclasses have the following properties:

Theorem 3.
(a) For class C(a) spaces, A_μ, B_μ, C_μ are gradient vectors, i.e.,

$$A_\mu = A_{,\mu}, \qquad B_\mu = B_{,\mu}, \qquad C_\mu = C_{,\mu}$$

(b) For class C(b) spaces, A_μ, B_μ, C_μ are hypersurface orthogonal vectors, i.e.,

$$A_\mu = aA_{,\mu}, \qquad B_\mu = bB_{,\mu}, \qquad C_\mu = cC_{,\mu}$$

(c) For class $C(c)$ spaces, A_μ, B_μ, C_μ have the form

$$A_\mu = aA_{,\mu} + \alpha_{,\mu}, \qquad B_\mu = bB_{,\mu} + \beta_{,\mu}, \qquad C_\mu = cC_{,\mu} + \gamma_{,\mu}$$

Proof of (c). Substitution of

$$pA_\mu + qB_\mu + rC_\mu = 0 \tag{23}$$

into (6) gives

$$3\Psi_0\{rA_{[\mu;\nu]}\} + \Psi_0\{rB_{[\mu;\nu]} + (2p+r)A_{[\mu}B_{\nu]}\} = 0 \tag{24}$$

$$\begin{aligned} &3\Psi_2\{rB_{[\mu;\nu]} - (2p+r)A_{[\mu}B_{\nu]}\} \\ &\quad + \Psi_0\{(r-4p)A_{[\mu;\nu]} - 4qB_{[\mu;\nu]} + (r_{,[\nu} - 4p_{,[\nu})A_{\mu]} \\ &\quad - 4q_{,[\nu}B_{\mu]}\} = 0 \end{aligned} \tag{25}$$

Putting

$$G_{\mu\nu} = (r_{,[\nu} - 4p_{,[\nu})A_{\mu]} + 4q_{,[\nu}B_{\mu]} \tag{26}$$

and noting

$$G_{[\mu\nu}A_\lambda B_{\rho]}=0 \tag{27}$$

$$G_{[\mu\nu}\{(r_{,\lambda}-4p_{,\lambda})A_{\rho]}-4q_{,\lambda}B_{\rho]}\}=0 \tag{28}$$

gives

$$3\Psi_2\{rA_{[\mu\nu}G_{\rho\lambda]}\}+\Psi_0\{rB_{[\mu;\nu}G_{\rho\lambda]}\}=0 \tag{29}$$

$$3\Psi_2\{rB_{[\mu;\nu}G_{\rho\lambda]}\}+\Psi_0\{(r-4p)A_{[\mu;\nu}G_{\rho\lambda]}-4qB_{[\mu;\nu}G_{\rho\lambda]}\}=0 \tag{30}$$

Since by definition, for C spaces,

$$4p\Psi_0{}^2-12q\Psi_0\Psi_2+r(9\Psi_2{}^2-\Psi_0{}^2)\neq 0 \tag{31}$$

then

$$A_{[\mu;\nu}G_{\rho\lambda]}=0=B_{[\mu;\nu}G_{\rho\lambda]} \tag{32}$$

Similarly one can show that

$$A_{[\mu;\nu}A_\rho B_{\lambda]}=0=B_{[\mu;\nu}A_\rho B_{\lambda]} \tag{33}$$

Using (32) and (33) in (24), one can show that

$$3\Psi_2\{rA_{[\mu;\nu}A_{\rho;\lambda]}\}+\Psi_0\{rB_{[\mu;\nu}A_{\rho;\lambda]}\}=0 \tag{34}$$

$$3\Psi_2\{rB_{[\mu;\nu}A_{\rho;\lambda]}\}+\Psi_0\{(r-4p)A_{[\mu;\nu}A_{\rho;\lambda]}-4qB_{[\mu;\nu}A_{\rho;\lambda]}\}=0 \tag{35}$$

and hence

$$A_{[\mu;\nu}B_{\lambda;\rho]}=0=A_{[\mu;\nu}A_{\lambda;\rho]} \tag{36}$$

Similarly it can be shown that

$$B_{[\mu;\nu}A_{\lambda;\rho]}=B_{[\mu;\nu}B_{\lambda;\rho]} \tag{37}$$

By Darboux's theorem one deduces the required form of A_μ and B_μ. ■

Proof of (*a*) *and* (*b*). These are just special cases of (c) and can be established by similar techniques. ■

In general for all class C spaces it is possible to establish scalar wave equations for Ψ_0, Ψ_2 by the same procedure as in the last section. However, by virtue of the above theorem, note that for class C spaces there are alternative scalar functions for which such equations can be established. Consider class C(a) as an example.

Class C(*a*) *Spaces* ($k^2>-3$). The relations (5) can be integrated to give

$$\begin{aligned}\Psi_2=\,&c_1\exp\{3A+[(k^2+3)^{1/2}+k]B\}\\&-c_2\exp\{3A-[(k^2+3)^{1/2}-k]B\}\end{aligned} \tag{38}$$

$$\begin{aligned}\Psi_0=\,&c_1[(k^2+3)^{1/2}+k]\exp\{3A+[(k^2+3)^{1/2}+k]B\}\\&+c_2[(k^2+3)^{1/2}-k]\exp\{3A-[(k^2+3)^{1/2}-k]B\}\end{aligned} \tag{39}$$

where c_1, c_2 are arbitrary constants. Now substituting in (4) gives

$$(k^2-1)B^{;\mu}B_{,\mu}=k\Psi_0-3\Psi_2 \tag{40}$$

$$A^{;\mu}{}_{\mu}+A^{,\mu}A_{,\mu}-B^{,\mu}B_{,\mu}=-2\Psi_2 \tag{41}$$

$$B^{;\mu}{}_{\mu}+2B^{,\mu}A_{,\mu}+2kB^{,\mu}B_{,\mu}=2\Psi_0 \tag{42}$$

where Ψ_0 and Ψ_2 are given by (38) and (39).

6. SUMMARY

I have presented a classification scheme which enables one to investigate Einstein–Petrov type I spaces in a systematic way, starting with spaces with comparatively simple field equations and progressing to spaces with less simplifying assumptions. It was important to discover that some of the spaces with the simplest field equations still had functionally independent Petrov scalars; and so such spaces would be fully representative of the algebraically general class. For these simpler spaces I showed that the field equations included three coupled differential equations for the Petrov scalars. These equations are the starting point for the integration procedure outlined in I; the results will be presented in another paper.

Finally, I note that the present integration scheme requires that Ψ_0 and Ψ_2 supply four real, functionally independent scalars, whereas the classification scheme picks out classes where these two complex scalars are functionally independent. This presents no problem, since in those class A spaces in which we are interested we are dealing directly with Ψ_0, Ψ_2 and so can always insist on the stronger condition directly; for Class C spaces we may deal with two other complex scalars, but providing we insist that their four real scalars are functionally independent, then it is easy to see that this ensures that the same condition holds for Ψ_0 and Ψ_2.

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